

# Not All Graphs are Pairwise Compatibility Graphs

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**Abstract.** Given an edge weighted tree  $T$  and two non-negative real numbers  $d_{min}$  and  $d_{max}$ , a *pairwise compatibility graph of  $T$  for  $d_{min}$  and  $d_{max}$*  is a graph  $G = (V, E)$ , where each vertex  $u \in V$  corresponds to a leaf  $u$  of  $T$  and there is an edge  $(u, v) \in E$  if and only if  $d_{min} \leq d_T(u, v) \leq d_{max}$  in  $T$ . Here,  $d_T(u, v)$  denotes the distance between  $u$  and  $v$  in  $T$ , which is the sum of the weights of the edges on the path from  $u$  to  $v$ . We call  $T$  a *pairwise compatibility tree* of  $G$ . We call a graph  $G$  a *pairwise compatibility graph (PCG)* if there exists an edge weighted tree  $T$  and two non-negative real numbers  $d_{min}$  and  $d_{max}$  such that  $G$  is a pairwise compatibility graph of  $T$  for  $d_{min}$  and  $d_{max}$ . Since the introduction of PCGs it remains an open problem: whether or not all undirected graphs are PCGs; in other words, is there always a pairwise compatibility tree  $T$  for any arbitrary graph  $G$ ? In this paper we give a negative answer to the open problem by showing that not all undirected graphs are PCGs.

## Introduction

Pairwise compatibility graphs have their origin in *Phylogenetics*, which is a branch of computational biology that concerns with reconstructing evolutionary relationships among organisms [1]. However, PCG's most intriguing potential lies in solving the *Clique Problem* in polynomial time [1], which is a well known NP-complete problem. Since their inception PCG have posed several interesting problems in front of us, and hitherto most of these problems have remained unsolved. Phillips showed that every graph on five vertices or less is a PCG [2] and Yanhaona *et al.* showed that all cycles and single chord cycles are PCGs [3]. But it is still unresolved whether or not every graph is a PCG.

## Results

The number of trees with  $n$  leaves is infinite. Moreover,  $d_{min}$  and  $d_{max}$  are any non-negative real numbers. Hence, there may be an exponential number of trees with suitable range of real numbers that can generate a particular graph as a PCG. This abundance of choices may give an impression that all graphs are pairwise compatibility graphs. The proponents of PCGs conceived that all undirected graphs are PCGs [1]. However, we have observed that regardless of the topology of  $T$  and the values of  $d_{min}$  and  $d_{max}$ , the placement of a vertex of a graph  $G$ , as a leaf in an edge weighted tree  $T$ , imposes constraints (in terms of relative distances) on the placements of the remaining vertices. As a result, in some cases, it becomes infeasible to satisfy all those constraints even when there are an exponential number of candidate trees. Before proceeding to our main result we now define some terms that we will use in this paper. A *subtree induced by a set of leaves* of  $T$  is the minimal subtree of  $T$  which contains the leaves. We represent a pairwise compatibility graph of  $T$  with  $d_{min}$  and  $d_{max}$  by  $PCG(T, d_{min}, d_{max})$ . Following theorem summarizes our result.

**Theorem 1** *Not all graphs are pairwise compatibility graphs.*

We have used following lemmas to prove Theorem 1.

**Lemma 1** *Let  $T$  be an edge weighted tree, and  $u, v$  and  $w$  be three leaves of  $T$  such that  $P_{uv}$  is the largest path in the subtree of  $T$  induced by  $u, v$  and  $w$ . Let  $x$  be a leaf of  $T$  other than  $u, v$  and  $w$ . Then in  $T$ ,  $d_T(w, x) \leq d_T(u, x)$  or  $d_T(w, x) \leq d_T(v, x)$ .*

**Lemma 2** *Let  $G = (V, E)$  be a  $PCG(T, d_{min}, d_{max})$ , and  $a, b, c, d$  and  $e$  be five vertices of  $V$ . Furthermore, let  $P_{ae}$  be the largest path in the subtree of  $T$  induced by the leaves representing  $a, b, c, d$  and  $e$ , and  $P_{bd}$  be the largest path in the subtree induced by the leaves representing  $b, c$  and  $d$ . Then  $V$  has no vertex  $x$  such that  $x$  is adjacent to  $a, c$  and  $e$  but not adjacent to  $b$  and  $d$ .*

**Lemma 3** *Let  $G = (V, E)$  be a graph having  $A = \{a_1, a_2, a_3, a_4, a_5\}$  and  $B = \{b_1, b_2, \dots, b_{10}\}$  as two disjoint subsets of vertices such that each vertex of  $B$  is adjacent to exactly three vertices of  $A$  and no two vertices of  $B$  has the same three neighbours in  $A$ . Then  $G$  is not a pairwise compatibility graph.*

## Conclusion

In this paper, we found a class of graphs that is not PCG and hence solved the open problem regarding whether or not every graph is a PCG. It would be interesting to recognize the classes of graphs that are PCGs.

## References

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